Math 451: Introduction to General Topology
Lecture 22

Prop. If X is a compact top. space and $Y \subseteq X$ is closed then Y is compact.

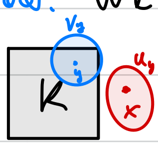
Proof. Let $\mathcal{U}$ be a cover of Y with sets open in X. Then $\mathcal{U} \cup \{Y^c\}$ is an open cover of X, so it has a finite subcover $\mathcal{V} \subseteq \mathcal{U} \cup \{Y^c\}$. Then $\mathcal{V} \cap \mathcal{U}$ is a finite subcover of Y since $\mathcal{V} \cap \mathcal{U} = \mathcal{V} \setminus \{Y^c\}$. $\square$

Counter-example to the converse. A compact subset of a top. space may not be closed.
For example, in the half-open top on $\{0,1\}$, i.e. $\mathcal{T} := \{\emptyset, \{0,1\}, \{1\}\}$, the set $\{1\}$ is compact (because it is finite) but it isn't closed. This top is not even T_1.
For a T_1 example, consider the cofinite top on $\mathbb{N}$. Then every set is compact, in particular $\mathbb{N} \setminus \{0\}$ is compact but it is not closed (proper closed sets are finite).

Although even T_1 is not enough for compact sets to be closed, T_2 is:

Prop. In a Hausdorff space X, every compact subset $K \subseteq X$ is closed.

Proof. We need to show that $X \setminus K$ is open and we do so by fixing an arbitrary $x \in X \setminus K$



and proving that $\exists$ open $U \ni x$ s.t. $U \cap K = \emptyset$. For each $y \in K$, by Hausdorffness, $\exists$ open $V_y \ni y$ and $U_y \ni x$ s.t. $V_y \cap U_y = \emptyset$. Then $\{V_y\}_{y \in K}$ is an open cover of K, hence $\exists$ finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$. But then $U := \bigcap_{i=1}^{n} U_i$ is disjoint from $\bigcup_{i=1}^{n} V_{y_i} \supseteq K$, so $U \cap K = \emptyset$, and U is open. $\square$

Cor. In metric spaces, compact sets are closed and bounded.

Counterexamples to the converse. One can always change a given metric d on X to another one $d' \leq 1$ generating the same topology; indeed, $d' := \min(d, 1)$. So in $\mathbb{R}$, $\mathbb{R}$ is not compact but it is closed and d'-bdd. Another natural example is the Baire space

$\mathbb{N}^{\mathbb{N}}$ with its usual metric d. By def, $d \leq 1$ and $\mathbb{N}^{\mathbb{N}}$ is closed in $\mathbb{N}^{\mathbb{N}}$ but not compact.

Continuous functions and compactness.

Recall that a function $f: X \to Y$ is continuous if $f^{-1}(\text{open}) = \text{open}$; equiv., $f^{-1}(\text{closed}) = \text{closed}$ (because f^{-1} commutes with complements).

Obs. Continuous functions map compact spaces to compact sets, i.e. if $f: X \to Y$ is continuous and X is compact then $f(X)$ is compact.

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Let $f^{-1}(\mathcal{V}) := \{f^{-1}(V) : V \in \mathcal{V}\}$. Then this is an open cover of X so it has a finite subcover $\{f^{-1}(V_1), f^{-1}(V_2), \dots, f^{-1}(V_n)\}$. Then $\{f(f^{-1}(V_i))\}_{i=1}^{n}$ covers $f(X)$ but each $f(f^{-1}(V_i)) \subseteq V_i$, so $\{V_1, \dots, V_n\}$ is a finite subcover of $f(X)$. $\square$

Cor (from calculus). Continuous functions $f: [a,b] \to \mathbb{R}$ attain their maximum and minimum.

Proof. We will prove below that $[a,b]$ for $a \leq b$ in $\mathbb{R}$ is compact. Then $f([a,b])$ is still a compact subset of $\mathbb{R}$, hence closed and bdd. Thus, it has sup and inf by bddness which are attained by closedness. $\square$

Cor. Let X be a compact top. space and Y a Hausdorff top space. Then:

(a) Every continuous $f: X \to Y$ maps closed sets to closed sets, i.e. if $C \subseteq X$ is closed, then $f(C)$ is also closed.

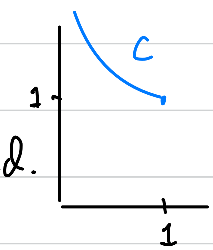
(b) Every continuous injection $f: X \hookrightarrow Y$ is actually an embedding, i.e. $f^{-1}: f(X) \to Y$ is automatically continuous. In particular, every continuous injection of $2^{\mathbb{N}}$ into a Hausdorff space is an embedding.

Proof. (a) Since $C \subseteq X$ is closed and X is compact, C is compact, so $f(C)$ is compact, hence closed because Y is Hausdorff.

(b) To check that $g := f^{-1}$ is continuous, one has to check that $g^{-1}(\text{closed}) = \text{closed}$. But $g^{-1} = (f^{-1})^{-1}$, so we need to check that $f(\text{closed}) = \text{closed}$, which is given by (a). □

Remark. The assumption of compactness in (a) and (b) is essential. For (b), we already saw this: the identity map from the discrete top. on $\mathbb{R}$ to $\mathbb{R}$ with the Euclidean top is a continuous bij but its inverse is not continuous.

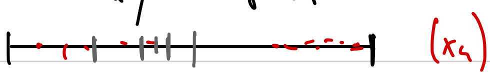
For (a), consider the set $C := \{(x,y) \in \mathbb{R}^2 : 0 < x \leq 1 \text{ and } y = \frac{1}{x}\}$. The map $\text{proj}_0 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, but $\text{proj}_0(C) = (0,1]$ is not closed.



Compactness for metric spaces.

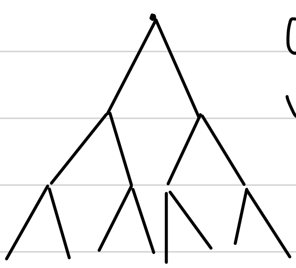
Def. A topological space is called *sequentially compact* if every sequence has a convergent subsequence.

Examples. (a) The Bolzano–Weierstrass theorem states that every bounded sequence in $\mathbb{R}$ has a convergent subsequence; equivalently, every closed and bdd subset of $\mathbb{R}$ is sequentially compact.

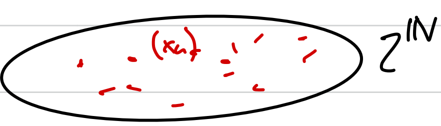
Proof. 

(b) The Cantor space $2^{\mathbb{N}}$ is sequentially compact. Same holds for $\Sigma^{\mathbb{N}}$ with $|\Sigma| < \infty$.

Proof. Let $(x_n) \subseteq 2^{\mathbb{N}}$. Call $w \in 2^{<\mathbb{N}}$ heavy if $[w]$ contains ∞-many members of (x_n).



Clearly the root $\emptyset$ is heavy, and pigeonhole principle, every heavy $w \in 2^{<\mathbb{N}}$ has a heavy child, either $w0$ or $w1$. Using this one obtains an ∞ branch $x \in 2^{\mathbb{N}}$ s.t. $x|_k$ is heavy for each $k \in \mathbb{N}$. Then we can choose a subseq. (x_{n_k}) s.t. $\forall \ell \geq k$ $x_{n_\ell} \in [x|_k]$, hence $\lim_{k \to \infty} x_{n_k} = x$. □



Unfortunately, sequential compactness and compactness are unrelated for general top spaces, i.e., neither implies the other.

Counterexample to "compactness ⇒ seq. compactness". By Tychonoff's theorem, arbitrary products of compact spaces are compact, so $[0,9]^{(0,1)}$ is compact in the product topology. However, it is not sequentially compact (because it's not 1^{st} ctbl). Indeed, let $f_n \in [0,9]^{(0,1)}$ be defined by $f_n : (0,1) \to [0,9] : x \mapsto$ the n^{th} digit in the decimal representation of $x = 0.x_0x_1x_2\ldots$, i.e. $f_n(x) := x_n$. We show that (f_n) doesn't have a pointwise convergent subsequence. Fix a subsequence $(f_{n_k})_{k\in\mathbb{N}}$. Let $x \in (0,1)$ be defined as follows: $x = 0.x_0x_1x_2\ldots$ where $x_n := \begin{cases} 1 & \text{if } n = n_{2k+1} \\ 0 & \text{otherwise} \end{cases}$. Then the sequence $(f_{n_k}(x)) = (0,1,0,1,\ldots)$ which doesn't converge in $[0,9]$.

Counterexample to "seq. comp ⇒ compactness". Let $(X,<)$ be a linearly ordered set, i.e. $<$ is a linear/total order on X, i.e. a binary relation on X which satisfies:

(i) $x \not< x$ for all $x \in X$.
(ii) $x < y \Rightarrow y \not< x$ for all $x,y \in X$.
(iii) $x<y$ and $y<z \Rightarrow x<z$ for all $x,y,z \in X$.

partial order

(iv) For each $x \neq y$ in X, $x<y$ or $y<x$.

The order topology on X is generated by the intervals $(a,b) := \{x \in X : a<x<b\}$. The set of these intervals is closed under finite intersections, hence it is a basis for the top.

Example. The usual top on $\mathbb{R}$ is the order topology with respect to the usual order $<$ on $\mathbb{R}$.

Now if X is an unctbl set equipped with a well-order $<$ (a total order in which every nonempty subset has a least element) s.t. every initial set $X_{<\alpha} := \{x \in X : x<\alpha\}$, for $\alpha \in X$, is ctbl, then the order top on X is 1^{st} ctbl, sequentially compact, but not compact. For example, the order $\in$ on the first unctbl ordinal ω_1.

Fortunately, seq. comp and compactness are equivalent for metric spaces, as we shall see.

Compactness for metric spaces.

We now define an important strengthening of bddness in metric spaces.

Def. A metric space (X, d) is called *totally bounded* if for each $\varepsilon > 0$ there is a *finite* cover of X with sets of diam $< \varepsilon$.

Obs. In the above definition, we can take the sets in the cover be open balls, i.e. (X, d) is totally bounded $\iff$ $\forall \varepsilon > 0$ $\exists$ finite cover of X with open balls of diam $< \varepsilon$.

Proof. If $\{P_1, \dots, P_n\}$ is a cover of X with diam $P_i < \varepsilon/2$, then choosing one $x_i \in P_i$, we see that $B_{\varepsilon/2}(x_i) \supseteq P_i$ hence $\{B_{\varepsilon/2}(x_i)\}_{i=1}^{n}$ is a finite cover of X with open balls of diam $< \varepsilon$. $\square$